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SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM OF ELASTICITY THEORY FOR AN INFINITE STRIP WITH SEVERAL SEMI-INFINITE SLITS

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A conformal mapping which does not belong to the class of rational mappings is used to solve the first fundamental problem of elasticity theory and permits finding the stress distribution in a strip with a different number of semi-infinite slits. Underlying the method is the use of dependences between the boundary values of the mapping function, which differ, in the general case, when the real axis is approached from the upper and lower half-planes. The solution obtained satisfies the boundary condition of the problem, whereupon the correctness of the solution is verified.

The domain D between lines bounding a strip and the edges of slits (Fig. 1) can be considered as the conformal transformation of the upper half-plane of the plane $\zeta = \sigma + i\eta$ onto the plane $z = x + iy$ by means of the function

$$z = c \ln \prod_{k=1}^{n} (a_k^2 - \zeta^2), \quad a_{k+1} > a_k \tag{1}$$

where $c > 0$, $a_k > 0$ are given constants.

In fact

$$x = c \ln \prod_{k=1}^{n} (\sigma^2 - a_k^2), \quad y = n\pi c$$

in the interval $(-\infty, -a_n)$ of the σ-axis since

$$\operatorname{Im} z = \sum_{k=1}^{n} \arg (a_k + \sigma)$$

When the point $\zeta = \sigma$ goes through the point $-a_n$, the vector $a_n + \zeta$ starting from the point $-a_n$ performs a rotation through an angle $-\pi$. Hence, in the interval $(-a_n, a_{n+1})$ where $a_{n-1} < |\sigma| < a_n$

$$x = c \ln (a_n^2 - \sigma^2) \prod_{k=1}^{n-1} (\sigma^2 - a_k^2), \quad y = (n-1)\pi c \tag{2}$$

where $-\infty < x < x_{n-1}$ and x_{n-1} equals the greatest value of the function $x = x(\sigma)$ defined by (2) in the interval $(-a_n, -a_{n-1})$. Therefore, the interval $(-a_n, -a_{n-1})$ of the σ-axis is transformed into a slit in the strip passing along the line $y = c(n-1)\pi$ from infinity to the point with abscissa $x = x_{n-1}$. It can be shown completely analogously that the interval $(-d_{n-1}, -d_{n-2})$ of the same axis is converted by means of the function (1) into a slit passing along the line $y = c(n-2)\pi$, where

$$x = c \ln (a_n^2 - \sigma^2)(a_{n-1}^2 - \sigma^2) \prod_{i=1}^{n-3} (\sigma^2 - a_i^2) \tag{3}$$

and $-\infty < x < x_{n-2}$, where x_{n-2} equals the greatest value of $x = x(\sigma)$ defined by (3).

It is similarly established that each of the intervals $(-a_n, -a_{n-1}), \ldots, (-a_1, a_1), \ldots, (a_{n-1}, a_n)$ of the σ - axis is converted into a slit along one of the lines parallel to the x - axis passing from infinity to one of the points with abscissas $x_{n-1}, \ldots, x_1$. The interval $(-a_1, a_1)$ is hence converted into a slit along the x - axis itself, which goes to the point $z = x_1$ where $x_1 = 2c \ln a_1 a_2 \ldots a_n$ is the image of the point $\zeta = 0$ according to (1). The abscissas $x_{n-3}, \ldots, x_2$ of the terminations of the remaining slits are found from the condition that each is the greatest value of the function $x = x(\sigma)$ defining the real part of z in the appropriate interval of the σ - axis. The imaginary part of z governing the equation of a line along which the slit passes will equal

$$y = \sum_{k=1}^{n} \arg (a_k^2 - \sigma^2)$$

When the point ζ passes each of the points $-a_n, \ldots, a_n$ the quantity y decreases by π taking on the values $y = \pi cl$, $l = 0, \pm 1, \ldots, \pm (n-1)$ (Fig. 1 corresponds to $n = 2$). Therefore, the whole real σ - axis is transferred into two infinite lines and a system of semi-infinite slits by means of (1). The terminations of these slits are pair-wise-symmetric relative to the σ - axis. Hence, points of the upper half-plane are transformed uniquely and conformally into interior points of the strips. Thus, for example, points of the η axis are transformed into the points

$$z = c \ln \prod_{i=1}^{n} (a_i^2 + \eta^2)$$

of the real ox axis to the right of the point $x = x_1$. The mapping inverse to (1) is not unique. However, by using some inequalities [1, 2], that branch which will be unique within the domain D can be extracted from the multivalued result of inverting the function (1). By using this branch and the function (1) the upper half-plane and the strip are mutually mapped uniquely and conformally.

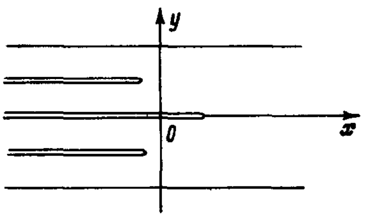

Fig. 1

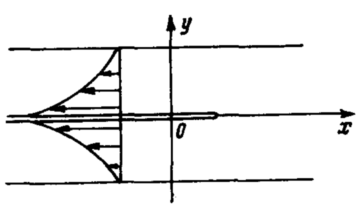

Fig. 2

Turning to the solution of the first fundamental problem using the transformation (1), let us first consider the case of a strip with one semi-infinite slit (Fig. 2). Then (1) becomes

$$z = c \ln (a^2 - \zeta^2) \tag{4}$$

A method for solving the fundamental problems of elasticity theory is given in the well-known monograph [3] for domains mapped on a half-plane by using rational functions. Let us show that use of transformations of the form (1) permits obtaining the solution of the problem mentioned for a strip with several slits or one slit, respectively, for different values of n and in the particular case (4) when $n = 1$, without belonging to the class of rational functions. The validity of the solution found is verified by the fact that it satisfies both the boundary condition and the solvability conditions for the problem. However, a number of specific singularities, which significantly complicate seeking the solution, will hence occur as compared to the use of rational mappings.

It is known [3 - 5] that the solution of the first fundamental problem for the domain mapped on the upper half-plane by means of the function $z = \omega(\zeta)$ reduces to determining two holomorphic functions $\Phi(\zeta)$ and $\Psi(\zeta)$ in the upper half-plane from the boundary condition (see [2], Sect. 92).

$$\Phi(\sigma) + \overline{\Phi(\sigma)} + \frac{1}{\overline{\omega'(\sigma)}}\{\overline{\omega(\sigma)}\,\Phi'(\sigma) + \omega'(\sigma)\,\Psi(\sigma)\} = N + iT \tag{5}$$

Here σ is a point on the real axis on the plane $\zeta = \sigma + i\eta$, and N and T are the normal and tangential stresses given on the edges of the strip and the edges of the slits as functions of the curvilinear coordinates introduced by the mapping (4).

In contrast to rational conformal mappings, the right side of (4) generally takes on different values on the real σ axis depending on whether ζ tends to σ from the upper or lower half-planes. Indeed, for $\zeta \to \sigma < -a$, remaining in the upper half-plane, then $z = c\ln|a^2 - \sigma^2| + i\pi c$; hence $\zeta \to \sigma < -a$, remaining in the lower half-plane, and $z = c\ln|a^2 - \sigma^2| - i\pi c$. The other intervals of the σ - axis are analogously mapped by means of (4).

Consequently, the σ - axis as a set of limit points satisfying the condition $\operatorname{Im}\zeta < 0$, $\zeta \to \sigma$, is transformed by means of (4) into the boundary lines of the strip determined by the formulas

$$z = \omega(\sigma) = c\,[\ln|\sigma^2 - a^2| + i\pi A_0], \qquad A_0 = \begin{cases} 1, & -\infty < \sigma < -a \\ 0, & -a < \sigma < a \\ -1, & a < \sigma < \infty \end{cases} \tag{6}$$

This axis as a set of limit points corresponding to the condition $\operatorname{Im}\overline{\zeta} < 0$, $\overline{\zeta} \to \sigma$ is transformed by means of (4) into the boundary lines of a strip defined by the formulas

$$z^* = \omega^*(\sigma) = c\,[\ln|\sigma^2 - a^2| - i\pi A_0] = \overline{\omega(\sigma)} \tag{7}$$

It follows from (6) and (7) that the upper and lower half-planes are transformed by means of (4) into two strips with slits, where one of the strips is the mapping of the other on the real x axis. Inserting $\omega'(\sigma) = -2c, \sigma/(a^2 - \sigma^2)$ into (5), the boundary condition of the problem can be written as

$$\begin{gathered} \sigma\Phi(\sigma) + \sigma\overline{\Phi(\sigma)} - \frac{1}{2c}\,\omega^*(\sigma)\,(a^2 - \sigma^2)\,\Phi'(\sigma) + \sigma\Psi(\sigma) = F \\ F = \sigma(N + iT) \end{gathered} \tag{8}$$

or going over to conjugate values in (8), as

$$\sigma\Phi(\sigma) + \sigma\overline{\Phi}(\sigma) - \frac{1}{2c}\,\omega(\sigma)\,(a^2 - \sigma^2)\,\overline{\Phi}'(\sigma) + \sigma\overline{\Psi}(\sigma) = \overline{F} \tag{9}$$

Expressing the condition that the function $\sigma\Psi(\sigma)$ defined by (8) is a boundary value of the function $\zeta\,\Psi(\zeta)$ which is holomorphic in the upper-half-plane and vanishes at infinity we obtain by using formula (21) of Sect. 76 in [3]

$$\frac{1}{2\pi i}\Big[\int_{-\infty}^{\infty}\frac{\overline{F}\,d\sigma}{\sigma - \zeta} - \int_{-\infty}^{\infty}\frac{\sigma\Phi(\sigma)}{\sigma - \zeta}\,d\sigma - \int_{-\infty}^{\infty}\frac{\sigma\overline{\Phi}(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2c}\,\frac{\omega(\sigma)\,(a^2 - \sigma^2)\,\overline{\Phi}'(\sigma)\,d\sigma}{\sigma - \zeta}\Big] = 0 \tag{10}$$

where ζ is a point in the upper half-plane. Since $\sigma\Phi(\sigma)$ $(\sigma\overline{\Phi}(\sigma))$ is the boundary value of the function $\zeta\Phi(\zeta)$ (or $\overline{\zeta}\,\overline{\Phi}(\zeta)$, respectively) which is holomorphic in the upper (lower) half-plane and vanishes at infinity, then the second integral equals $\zeta\Phi(\zeta)$ and the third is zero. To evaluate the fourth integral, the factors in the integrand must be reduced to a common domain of definition. To do this, it is sufficient to use the expression for $\omega(\sigma)$ in terms of $\omega^*(\sigma)$, which follows from (6) and (7)

$$\omega(\sigma) = \omega^*(\sigma) + i2\pi c A_0.$$

After this, the fourth integral in (4) can be written as

$$\frac{1}{4\pi ci}\int_{-\infty}^{\infty}\omega(\sigma)\chi(\sigma)\,d\sigma = \frac{1}{2\pi i}\left[\frac{1}{2c}\int_{-\infty}^{\infty}\overline{\omega}(\sigma)\chi(\sigma)\,d\sigma + \pi i\int_{-\infty}^{-a}\chi(\sigma)\,d\sigma - \pi i\int_{a}^{\infty}\chi(\sigma)\,d\sigma\right] \qquad (11)$$

$$\chi(\sigma) = \frac{(a^2-\sigma^2)\,\Phi'(\sigma)}{\sigma-\zeta}$$

Let us show that each of the integrals in the right side of (11) are zero. Indeed, noting that the integrand in the first integral has a point of discontinuity at $\sigma = \pm a$, we replace the line of integration by line L consisting of segments of the σ axis and semicircles of the small radius ρ described from the points a and $-a$ (Fig.3).

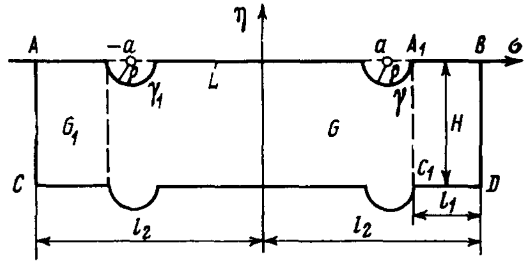

Fig.3

Then the integrand as a function of the variable point σ of the lower half-plane is holomorphic in the domain lying below the line L. Constructing the curvilinear rectangle $ABDC$ with two congruent sides H and $2l_2$, as shown in Fig.3, bounding the domain G, we have for all finite values of H and l_2

$$\int_R \omega(\sigma)\chi(\sigma)\,d\sigma = 2\pi i\sum \operatorname{res}(\zeta) = 0$$

where R is the contour of the domain G. The value obtained for the integral is independent of the magnitude of the perimeter R, therefore

$$\int_L \omega(\sigma)\chi(\sigma)\,d\sigma = \lim\int_R \omega(\sigma)\chi(\sigma)\,d\sigma = 0 \quad \text{for} \quad H\to 0, \quad l_2\to\infty$$

We see that the second and also the third integral are zero, respectively, from analogous calculations around the contour of the figure A_1BDC_1, in the right sides of (11) in the intervals $(-\infty, -(a+\rho))$, $((a+\rho), \infty)$.

Now, let us show that as $\rho\to 0$ the magnitudes of the integrals considered also remain zero. For the former, we have on an arc γ of a circle of radius ρ (Fig.3)

$$\left|\int_{\gamma}\frac{(a^2-\bar{\zeta}^2)\,c\ln(a^2-\bar{\zeta}^2)\,\overline{\Phi}'(\bar{\zeta})}{\bar{\zeta}-\zeta}\,d\bar{\zeta}\right|\leqslant 2a\pi c\rho\,|\ln\rho\,(2a+\rho)|\,\rho\,(2a+\rho)\,|\overline{\Phi}'(\bar{\zeta})|\,\frac{1}{m}$$

where m is a proportionality factor between ρ and $|\bar{\zeta}-\zeta|$ since $|\bar{\zeta}-\zeta|=m\rho$, $0<\rho<2$ (Fig. 3). The relation $|a^2-\bar{\zeta}^2|=|a-\bar{\zeta}||a+\bar{\zeta}|\leqslant\rho\,(2a+\rho)$ is used here. Since

$$\lim_{\rho\to 0}\rho^2\,|\ln\rho\,(2a+\rho)|=-\lim_{\rho\to 0}\frac{(a+\rho)\,\rho^2}{(2a+\rho)}=0$$

and $\Phi(\bar{\zeta})$ is a bounded function, then the right side of the estimate of the integral vanishes as $\rho\to 0$. It hence follows that the integral over the arc equals zero as $\rho\to 0$, which means that as the line L is rectified into the line $O\sigma$ the integral

$$\int_{-\infty}^{\infty}\omega(\sigma)\,\chi(\sigma)\,d\sigma=0$$

That the two other integrals are zero as $\rho\to 0$ is established analogously.

Therefore, we finally obtain from condition (10)

$$\Phi(\zeta)=\frac{1}{2\pi i\zeta}\int_{-\infty}^{\infty}\frac{F\,d\sigma}{\sigma-\zeta} \tag{12}$$

After this, the function $\zeta\,\Psi(\zeta)$ is determined by means of its boundary value from the formula (8)

$$\zeta\Phi(\zeta)=\frac{1}{2\pi i}\left[\int_{-\infty}^{\infty}\frac{F\,d\sigma}{\sigma-\zeta}-\int_{-\infty}^{\infty}\frac{\sigma\Phi(\sigma)\,d\sigma}{\sigma-\zeta}+\frac{1}{2c}\int_{-\infty}^{\infty}\frac{\omega^*(\sigma)\,(a^2-\sigma^2)}{\sigma-\zeta}\times\right. \tag{13}$$

$$\left.\Phi'(\sigma)\,d\sigma-\int_{-\infty}^{\infty}\frac{\sigma\overline{\Phi}(\sigma)\,d\sigma}{\sigma-\zeta}\right]$$

To evaluate the third integral on the right side of (13) we express $\omega^*(\sigma)$ in terms of $\omega(\sigma)$ on the basis of (6) and (7). It hence follows that

$$\omega^*(\sigma)=\omega(\sigma)-i2\,\pi cA_0 \tag{14}$$

Then the integral mentioned can be represented by the sum of integrals

$$\frac{1}{4\pi ci}\int_{-\infty}^{\infty}\omega^*(\sigma)\,\chi_1(\sigma)\,d\sigma=\frac{1}{4\pi ci}\left[-2\pi ci\int_{-\infty}^{-a}\chi_1(\sigma)\,d\sigma+\right. \tag{15}$$

$$\left.2c\pi i\int_{a}^{\infty}\chi_1(\sigma)\,d\sigma+\int_{-\infty}^{\infty}\omega(\sigma)\,\chi_1(\sigma)\,d\sigma\right]$$

$$\chi_1(\sigma)=\frac{a^2-\sigma^2}{\sigma-\zeta}\,\Phi'(\sigma)$$

To evaluate the first of the integrals in the right side of (15), we find first

$$J_1=\frac{-1}{2}\int_{C_1}\chi_1(\sigma)\,d\sigma$$

where G_1 is a rectangular domain: $A\leqslant\sigma\leqslant-(a+\rho)$, $-H\leqslant\eta\leqslant 0$ (Fig. 3), containing the point $\zeta: J_1=\Sigma\ \mathrm{res}\,(\zeta)=-\pi i\,(a^2-\zeta^2)\,\Phi'(\zeta)$. Since J_1 is independent of the size of the domain G_1 then we obtain for H, $\rho\to 0$, $A\to-\infty$

$$-\frac{1}{2}\int_{-\infty}^{-a}\chi(\sigma)\,d\sigma = -\pi i\,(a^2-\zeta^2)\,\Phi'(\zeta)$$

We find analogously for the second integral in the right side of (15)

$$\frac{1}{2}\int_{a}^{\infty}\chi_1(\sigma)\,d\sigma = \pi i\,(a^2-\zeta^2)\,\Phi'(\zeta)$$

The result of evaluating these integrals can be inserted into (13) in the form of one term $\Phi_1(\zeta)$ by setting

$$\Phi_1(\zeta) = -\,i\pi A_0\,(a^2-\zeta^2)\,\Phi'(\zeta) \tag{16}$$

Finally, let us note that the product $(a^2 - \sigma^2)\ \omega\ (\sigma) = (a^2-\sigma^2)\,c\ln(a^2 - \sigma^2)$ on the σ - axis has an eliminable discontinuity at $\sigma = \pm a$. Hence, the integrand of the integral

$$J_2 = \int_{-\infty}^{\infty}\frac{a^2-\sigma^2}{\sigma-\zeta}\,\Phi\ (\sigma)\ln(a^2-\sigma^2)\,d\sigma$$

is holomorphic in the upper half-plane and vanishes at infinity (since $\Phi'(\zeta)$ possesses this property by assumption), consequently $J_2 = (a^2-\sigma^2)\ln(a^2-\zeta^2)\,\Phi'(\zeta)$. Therefore, according to (16)

$$\Psi(\zeta) = \frac{1}{2\pi i\zeta}\int_{-\infty}^{\infty}\frac{F\,d\sigma}{\sigma-\zeta} - \Phi(\zeta) + \frac{a^2-\zeta^2}{2\zeta}\ln(a^2-\zeta^2)\,\Phi'(\zeta) + \frac{1}{\zeta}\,\Phi_1(\zeta) \tag{17}$$

The functions $\Phi\ (\zeta)$ and $\Psi\ (\zeta)$ are determined analogously for strips with any number of slits in conformity with the value of n in (1). Thus, for a strip with three slits (Fig. 1), we obtain after all the computations

$$\Phi(\zeta) = \frac{1}{2\pi i f(\zeta)}\int_{-\infty}^{\infty}\frac{\overline{F}\,d\sigma}{\sigma-\zeta}$$

$$\Psi(\zeta) = \frac{-1}{2\pi i f(\zeta)}\int_{-\infty}^{\infty}\frac{F\,d\sigma}{\sigma-\zeta} + \Phi(\zeta) -$$

$$\frac{(a_1^2-\zeta^2)(a_2^2-\zeta^2)}{2f(\zeta)}\ln\left[(a_1^2-\zeta^2)(a_2^2-\zeta^2)\right]\Phi'(\zeta) +$$

$$\frac{2\pi i\,(a_1^2-\zeta^2)(a_2^2-\zeta^2)}{f(\zeta)}\,\Phi'(\zeta)\,A_{12}$$

$$A_{12} = \begin{cases}1, & -\infty < \sigma < -a_2;\\ {}^1/_2, & -a_2 < \sigma < -a_1;\end{cases}\qquad A_{12} = 0,\ \ -a_1 < \sigma < a_1;$$

$$A_{12} = \begin{cases}-{}^1/_2, & a_1 < \sigma < a_2\\ -1, & a_2 < \sigma < \infty\end{cases}$$

Table 1

y	$\zeta=\sigma+i\eta$	φ	φ_1	φ_2	θ	X_x-Y_y
0^+	$a/2$	0	π	0	π	$-2p$
0^-	$-a/2$	π	π	0	π	$-2p$
π	$1,3a$	0	0	0	0	0
$-\pi$	$-1,3a$	π	π	π	0	0
$-\pi/2$	λa	18,5°	57°	10,5°	46,5°	$-0,156\,p$
$\pi/2$	$-\bar{\lambda}a$	161,5°	123°	169,5°	46,5°	$-0,156\,p$

Let us present an example of computing the stresses by means of (12) and (17) for the slit edges (Fig. 2) loaded by normal pressure

$$N(x) = -p\sigma(x) = \begin{cases} p\sqrt{a^2-e^{x/c}} & \text{for } -\infty < x^+ < x_1; \\ -p\sqrt{a^2-e^{x/c}} & \text{for } -\infty < x^- < x_1, \end{cases} \quad p = \text{const} > 0$$

$$\Phi(\zeta) = \frac{ip}{\pi}\left[\frac{a}{\zeta} + \frac{1}{2}\ln\frac{a-\zeta}{a+\zeta}\right] \quad \text{on} \quad (-a, a) \tag{18}$$

$$\Phi'(\zeta) = \frac{ip}{\pi}\left[\frac{2a}{\zeta^2-a^2} - \frac{a}{\zeta^2}\right]$$

$$\Psi(\zeta) = \frac{ipa}{\pi\zeta}\left[1 - \frac{1}{2}\left(\frac{a^2}{\zeta^2}-1\right)\right]\ln(a^2-\zeta^2) + \frac{1}{\zeta}\Phi_1(\zeta)$$

$$\Phi_1(\zeta) = ap\left(3-\frac{a^2}{\zeta^2}\right)A_0, \quad -\infty < \sigma < -a$$

Since $\operatorname{Re}\Phi(\zeta) = P/\pi(\sin\varphi - \theta/2)$, where $\varphi = \arg\zeta$, $\theta = \varphi_1 - \varphi_2$, $\varphi_1 = \arg(\zeta - a)$, $\varphi_2 = \arg(\zeta + a)$, then to construct the diagram (Fig. 2) of the distribution of values of the sum $X_x + Y_y$ over the section $x = 2\ln(\sqrt{3}a/2)$, we obtain the data presented below ($\lambda = 1.28 + 0.43i$).

Let us show that the solution governed by the functions (18) satisfies the boundary condition (9). In fact, for $\zeta = \sigma$

$$\Phi(\sigma) = -\frac{p}{\pi}\left[\left(\frac{1}{2}\ln\frac{\rho_1}{\rho_2} \pm \frac{a}{\rho}\right)i - \frac{\pi}{2}\right] \quad \text{on} \quad (-a, a)$$

$\rho = |\zeta|$; $\rho_1 = |\zeta - a|$, $\rho_2 = |\zeta + a|$, $\overline{\Phi}(\sigma) = P/\pi[-(1/2\ln\rho_1/\rho_2 \pm a/\rho)i - \pi/2]$

whereupon

$$\sigma[\Phi(\sigma) + \overline{\Phi}(\sigma)] = -\sigma p \tag{19}$$

in the same interval. Hence

$$-\frac{1}{2c}\omega(\sigma)(a^2-\delta^2)\overline{\Phi}'(\sigma) = \frac{ipa}{2\pi}\left(3-\frac{a^2}{\sigma^2}\right)\ln(a^2-\sigma^2) \tag{20}$$

$$\sigma\overline{\Psi}(\sigma) = -\frac{ipa}{2\pi}\left(3-\frac{a^2}{\sigma^2}\right)\ln(a^2-\sigma^2) \tag{21}$$

The boundary condition (9), as the sum of (19) - (21), goes over into a given quantity $N(\sigma) = -p\sigma$ in the interval $(-a, a)$, and as is easy to compute, vanishes outside this interval, which is a confirmation of the correctness of the solution.

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